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QUALITATIVE PROPERTIES OF THE ERLANG BLOCKING MODEL WITH HETEROGENEOUS USER REQUIREMENTS

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Propriétés Qualitatives du Modèle d'Erlang en Présence de Trafics Hétérogènes

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Résumé

Nous étudions l'impact sur les performances d'une modification des paramètres (fréquence des appels, intensité des trafics, etc.) dans un modèle d'Erlang généralisé. Des propriétés de monotonicité et de concavité sont mises en évidence pour les principales caractéristiques du modèle (probabilités de congestion, débit des canaux, etc.), à la fois en régime transitoire et en régime stationnaire. Ces résultats sont obtenus en utilisant des méthodes de comparaison et de couplage stochastiques.

Mots-Clés: Intégration de services, réseaux à commutation de circuits, évaluation des performances, couplage, ordre stochastique, monotonicité, concavité.

Qualitative Properties of the Erlang Blocking Model with Heterogeneous User Requirements

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Abstract

We study the effect of increasing the model parameters (e.g. arrival rates and traffic intensities) in the Erlang blocking model with heterogeous user requirements. First-order (monotonicity) and second-order (concavity) qualitative results are obtained for the performance measures of interest (loss probabilities, throughput, channel occupancy, etc.) both in the transient and in the steady-state cases. Stochastic and likelihood-ratio orderings together with coupling techniques are used to indicate the effect of modifying the model parameters.

Keywords: Service integration, circuit-switched network, performance evaluation, coupling, stochastic ordering, monotonicity, concavity.

1 Introduction

We consider the classical Erlang blocking model with heterogeneous user requirements. We suppose that K types of traffic or calls arrive at a switching center according to K independent Poisson processes with state-dependent rates $\lambda_k(n_k)$, where n_k denotes the number of type-k transmissions in progress, $k = 1, 2, \ldots, K$. The switching center has N outgoing channels (see Figure 1). On arrival, a type-k call requires b_k channels. If there are fewer than b_k free channels the call is lost, otherwise b_k channels are immediately seized for a random time. When n type-k transmissions are in progress, the type-k call-completion rate is $\mu_k(n)$. The b_k channels are arbitrarily chosen among the free channels and are simultaneously released at the end of the transmission period. For each type of call, we assume that the holding times form a sequence of i.i.d. random variables with

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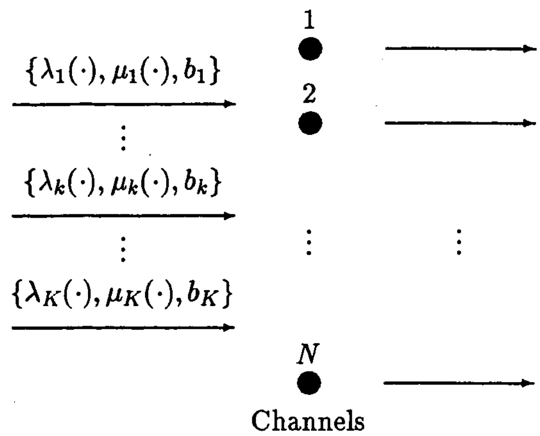

Figure 1: Switching center

an arbitrary common distribution. We further assume that the call holding-times are mutually independent and are independent of the arrival processes.

The equilibrium behavior of Erlang-like traffic models has received considerable attention in the past where both the analytical (e.g. [1], [2], [3], [15]) and the numerical aspects (e.g. [4], [7], [8], [10], [16]) have been investigated. These papers provide closed-form expressions and numerical procedures for computing the basic performance measures. Some of these results are recalled below.

Until recently, no satisfactory expressions for the transient behavior had been reported. Part of this gap has been filled by Mitra and Weiss [9] who have used the theory of large deviations to give an asymptotic analysis of the probability of blocking at time t assuming that all channels are initially idle (resp. busy). Simonian [14] has recently carried out a very detailed asymptotic analysis of the channel occupancy (see also [1]).

The approach we take in this paper is different from prior work since our objective is to study the effect of increasing the model parameters on the performance measures of interest. More precisely, we focus on *monotonicity* properties and on *concavity* properties of the main performance measures (loss probabilities, throughput, channel occupancy, etc.), both in the transient and in the steady-state cases.

This study is motivated by the ever-growing interest in circuit-switched networks that support different services with different characteristics (e.g. different bandwidth requirements, arrival rates and call holding-times for services such as telephone, video and facsimile).

For future reference, we now recall some well-known equilibrium results of the Erlang traffic model.

Let $\mu_k(n) = n\mu_k$ for $k = 1, 2, \ldots, K$, $n \geq 0$ (constant service rates) and let X_k be the number of type-k calls in progress in steady-state. We have [7]

$$\mathrm{P}(\mathbf{X}=\mathbf{n}) = G^{-1}(N)\prod_{k=1}^{K}\left\{\frac{\prod_{l=0}^{n_k-1}\lambda_k(l)}{n_k!\mu_k^{n_k}}\right\}, \tag{1.1}$$

for $\mathbf{b}\cdot\mathbf{n} \le N$, where $\mathbf{X} := (X_1, X_2, \ldots, X_K)$, $\mathbf{n} := (n_1, n_2, \ldots, n_K)$ and $\mathbf{b} := (b_1, b_2, \ldots, b_K)$.

The normalization constant $G(N)$ in (1.1) is given by

$$G(N) = \sum_{\mathbf{b}\cdot\mathbf{n}\le N}\prod_{k=1}^{K}\left\{\frac{\prod_{l=0}^{n_k-1}\lambda_k(l)}{n_k!\,\mu_k^{n_k}}\right\}. \tag{1.2}$$

Denote by

$$Z := \sum_{k=1}^{K} b_k X_k, \tag{1.3}$$

the *total* number of calls in progress in steady-state. Also define β_k and T_k as the *loss* probability and the *throughput* for type-k calls, respectively, $k = 1, 2, \ldots, K$.

Clearly (cf. (1.1)-(1.3))

$$\beta_k = \mathrm{P}(Z > N - b_k) = 1 - \frac{G(N-b_k)}{G(N)}; \tag{1.4}$$

$$T_k = \mu_k \mathrm{E}(X_k), \tag{1.5}$$

for $k = 1, 2, \ldots, K$.

If we further assume that $\lambda_k(\cdot) \equiv \lambda_k$ for $k = 1, 2 \ldots, K$ (constant arrival rates), then

$$T_k = \lambda_k(1-\beta_k) \tag{1.6}$$

$$= \lambda_k \frac{G(N-b_k)}{G(N)}, \qquad \text{for } k = 1, 2, \ldots, K, \tag{1.7}$$

from (1.4).

These results show, in particular, that the main steady-state performance measures can be expressed in terms of the normalization constant $G(N)$. However, because of the shape of $G(N)$, a straightforward derivation of monotonicity/concavity results is not feasible in most cases.

In Section 2, a coupling technique is employed as in [17] to derive time-dependent monotonicity results for $K = 2$. In particular, we show that the number of type-1 (resp. type-2) transmissions in progress at time t is *stochastically* increasing (resp. decreasing) with respect to the vector arrival rates of the type-1 calls. Section 3 addresses the equilibrium behavior. In particular, we show that for $K = 2$ and $b_1 \leq b_2$, Z is increasing as a function of λ_2 in the sense of *likelihood-ratio ordering* if and only if certain conditions on b_1 and b_2 prevail. Additional results are also obtained for $K > 2$ (more results in the case $K > 2$ can be found in [11] where a different approach is followed). In Section 4 we study concavity properties of the original Erlang model (i.e. $K = 1$). We show that $\mathrm{E}(f(X(t))$, where $X(t)$ denotes the number of busy channels at time t, is a concave increasing function of the arrival rate for all increasing mappings $f : \mathbb{R} \to \mathbb{R}$ and for all $t > 0$.

2 Monotonicity: Time-Dependent Analysis

We assume throughout this section that $K = 2$. A mapping $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *increasing* (resp. *decreasing*) if $f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$) for all $x, y \in \mathbb{R}^n$ such that $x < y$, where the last inequality is interpreted componentwise whenever $n \geq 2$.

We first introduce some notation and definitions. For $k = 1, 2$, define:

- $\underline{\lambda}_k := (\lambda_k(0), \lambda_k(1), \ldots, \lambda_k(\lfloor N/b_k \rfloor - 1))$;
- $\underline{\mu}_k := (\mu_k(1), \mu_k(2), \ldots, \mu_k(\lfloor N/b_k \rfloor))$,

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. We shall use the symbol $[\underline{\lambda}_1, \underline{\lambda}_2]$ to identify the system when the vector arrival rates are $\underline{\lambda}_1$ and $\underline{\lambda}_2$ (the other model parameters $\underline{\mu}_1$, $\underline{\mu}_2$, N and $\mathbf{b}$ will be kept constant, so there will be no possible confusion with this notation). We replace $\underline{\lambda}_k$ by λ_k in the above notation when the type-k call arrival rate is constant and equal to λ_k, $k = 1, 2$. For $k = 1, 2$, and $t \geq 0$, we define:

- $X_k(t)$ as the number of type-k transmissions in progress at time t;
- $B_k(t)$ as the number of type-k calls that have been lost by time t;
- $D_k(t)$ as the number of completed type-k transmissions by time t.

Definition 2.1 Let $U(\mathbf{v})$ be a real-valued random variable parameterized by some vector $\mathbf{v} \in \mathbb{R}^n$, $n \geq 1$. $U(\mathbf{v})$ is said to be *stochastically increasing in* $\mathbf{v}$ if

$$\mathrm{E}(f(U(\mathbf{v}))) \leq \mathrm{E}(f(U(\mathbf{w})))$$

for all increasing mappings $f : \mathbb{R} \to \mathbb{R}$ and all $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v} \leq \mathbf{w}$, where the last inequality is interpreted componentwise whenever $n \geq 2$. The notation $U(\mathbf{v}) \leq_{st} U(\mathbf{w})$ will be used.

We have the following theorem:

Theorem 2.1 *Assume that $\lambda_1(\cdot)$, $\mu_1(\cdot)$ and $\mu_2(\cdot)$ are increasing functions. Assume also that $\lambda_2(n) := \lambda_2 > 0$ for $n = 0, 1, \ldots, \lfloor N/b_2 \rfloor - 1$ (constant arrival rate for type-2 calls). If the call holding-times are exponentially distributed, then for all $t > 0$,*

1. *$X_1(t)$, $B_2(t)$ and $D_1(t)$ are stochastically increasing in $\underline{\lambda}_1$;*
2. *$X_2(t)$ and $D_2(t)$ are stochastically decreasing in $\underline{\lambda}_1$.*

The same results holds when indices 1 and 2 are interchanged.

Proof. The proof is based on a coupling argument as in [17]. Fix $\underline{\lambda}'_1 := (\lambda'_1(0), \ldots, \lambda'_1(\lfloor N/b_1 \rfloor - 1))$ such that $\underline{\lambda}_1 \leq \underline{\lambda}'_1$. We construct four birth-death processes $\mathbf{Y}_k := \{Y_k(t),\, t \geq 0\}$ and $\mathbf{Y}'_k := \{Y'_k(t),\, t \geq 0\}$ $(k = 1, 2)$ on a *common probability space* by generating transitions using a single Poisson process with rate

$$\gamma := \max_n \lambda'_1(n) + \lambda_2 + [\mu_1(\lfloor N/b_1 \rfloor) + \mu_2(\lfloor N/b_2 \rfloor)]\, N. \tag{2.1}$$

Suppose a point in this Poisson process occurs at time t. Then, at time t,

(T1) with probability $\lambda_1(Y_1(t-))/\gamma$ a birth occurs

- in $\mathbf{Y}_1$ if $b_1 Y_1(t-) + b_2 Y_2(t-) \leq N - b_1$,
- in $\mathbf{Y}'_1$ if $b_1 Y'_1(t-) + b_2 Y'_2(t-) \leq N - b_1$;

(T2) with probability $[\lambda'_1(Y'_1(t-)) - \lambda_1(Y_1(t-))]\,/\gamma$ a birth occurs

- in $\mathbf{Y}'_1$ if $b_1 Y'_1(t-) + b_2 Y'_2(t-) \leq N - b_1$;

(T3) with probability λ_2/γ a birth occurs

- in $\mathbf{Y}_2$ if $b_1 Y_1(t-) + b_2 Y_2(t-) \leq N - b_2$,
- in $\mathbf{Y}'_2$ if $b_1 Y'_1(t-) + b_2 Y'_2(t-) \leq N - b_2$;

(T4) with probability $\mu_1(Y_1(t-))/\gamma$ a death occurs in $\mathbf{Y}_1$ *and* in $\mathbf{Y}'_1$;

(T5) with probability $[\mu_1(Y'_1(t-)) - \mu_1(Y_1(t-))]\,/\gamma$ a death occurs in $\mathbf{Y}'_1$;

(T6) with probability $\mu_2(Y'_2(t-))/\gamma$ a death occurs in $\mathbf{Y}_2$ *and* in $\mathbf{Y}'_2$;

(T7) with probability $[\mu_2(Y_2(t-)) - \mu_2(Y'_2(t-))]\,/\gamma$ a death occurs in $\mathbf{Y}_2$;

(T8) with probability $1 - [\lambda'_1(Y'_1(t-)) + \lambda_2 + \mu_1(Y'_1(t-)) + \mu_2(Y_2(t-))]\,/\gamma$ no event occurs in $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}'_1$, $\mathbf{Y}'_2$.

The above construction, together with the initial conditions $Y_k(0) = Y_k'(0) := y_k$ ($k = 1, 2$), implies the following inequalities:

$$Y_1(t) \leq Y_1'(t); \tag{2.2}$$
$$Y_2(t) \geq Y_2'(t), \tag{2.3}$$

for all $t \geq 0$.

Before proving (2.2) and (2.3) it should be noted that these inequalities together with the assumptions on the monotonicity of the functions $\lambda_1(\cdot)$, $\mu_1(\cdot)$ and $\mu_2(\cdot)$ fully justify the validity of transitions (T2), (T5) and (T7).

The proof of (2.2)-(2.3) proceeds by induction. Consider the transition (T1). We first notice that both inequalities hold for $t = 0$. Let $t > 0$ be a point of the Poisson process and assume that (2.2)-(2.3) hold in $[0, t)$. If $Y_1(t-) < Y_1'(t-)$ then clearly $Y_1(t) \leq Y_1'(t)$ since there is *at most* one birth in $\mathbf{Y}_1$ and in $\mathbf{Y}_1'$ at time t. If $Y_1(t-) = Y_1'(t-)$ then

$$b_1 Y_1'(t-) + b_2 Y_2'(t-) \leq b_1 Y_1(t-) + b_2 Y_2(t-), \tag{2.4}$$

from the induction hypothesis (2.3). So, either $Y_1(t) = Y_1'(t)$ (this is the case if $b_1 Y_1(t-) + b_2 Y_2(t-) \leq N - b_1$ or if $b_1 Y_1'(t-) + b_2 Y_2'(t-) > N - b_1$) or $Y_1(t) < Y_1'(t)$ (this is the case if $b_1 Y_1'(t-) + b_2 Y_2'(t-) \leq N - b_1 < b_1 Y_1(t-) + b_2 Y_2(t-)$).

On the other hand, the induction hypothesis $Y_2(t-) \geq Y_2'(t-)$ entails $Y_2(t) \geq Y_2'(t)$ since no event occurs at time t in $\mathbf{Y}_2$ and $\mathbf{Y}_2'$ under a transition of type (T1). The remaining "birth-transitions" (T2) and (T3) are treated similarly.

Consider now (for instance) the transition (T5). If $Y_1(t-) = Y_1'(t-)$ then (T5) cannot occur at time t (the probability of this event is zero) and so $Y_1(t) = Y_1'(t)$. If $Y_1(t-) < Y_1'(t-)$ then clearly $Y_1(t) \leq Y_1'(t)$ since there is only one departure from $\mathbf{Y}_1'$.

Finally, $Y_2(t-) \geq Y_2'(t-)$ implies $Y_2(t) \geq Y_2'(t)$ since $\mathbf{Y}_2$ and $\mathbf{Y}_2'$ are not modified under a transition of type (T5). The remaining "death-transitions" (T4), (T6) and (T7) are treated similarly.

We now use the previous results as follows. First, we observe that the above construction yields the following key properties:

- $\mathbf{Y}_1$ (resp. $\mathbf{Y}_2$) has the same distribution as $\{X_1(t),\, t \geq 0\}$ (resp. $\{X_2(t),\, t \geq 0\}$) in the system $[\underline{\lambda}_1, \lambda_2]$;

- $\mathbf{Y}_1'$ (resp. $\mathbf{Y}_2'$) has the same distribution as $\{X_1'(t),\, t \geq 0\}$ (resp. $\{X_2'(t),\, t \geq 0\}$), where $X_k'(t)$ denotes the number of type-k calls in progress at time t in the system $[\underline{\lambda}_1', \lambda_2]$, $k = 1, 2$.

Secondly, we apply the so-called *coupling theorem* of Kamae, Krengel and O'Brien [5, Theorem 1], which gives us (cf. (2.2)-(2.3))

$$\begin{aligned} X_1(t) &\leq_{st} X_1'(t); \\ X_2(t) &\geq_{st} X_2'(t), \end{aligned}$$

for all $t \geq 0$. This proves the first part of statements 1 and 2.

Consider now $D_k(t)$, $k = 1, 2$. Let

- $E_k(t)$ (resp. $E_k'(t)$) be the number of deaths in $\mathbf{Y}_k$ (resp. $\mathbf{Y}_k'$) by time t, $k = 1, 2$.

We observe from transitions (T4)-(T5) and (T6)-(T7) respectively, that

$$E_1(t) \leq E_1'(t); \tag{2.5}$$

$$E_2(t) \geq E_2'(t), \tag{2.6}$$

for all $t \geq 0$.

Since $\{E_k(t),\, t \geq 0\}$ has the same distribution as $\{D_k(t)\, t \geq 0\}$ in $[\underline{\lambda}_1, \lambda_2]$, and $\{E_k'(t),\, t \geq 0\}$ has the same distribution as $\{D_k'(t)\, t \geq 0\}$, where $D_k'(t)$ is the number of type-k calls that have been completed by time t in $[\underline{\lambda}_1', \lambda_2]$ $(k = 1, 2)$, we deduce from (2.5) (resp. (2.6)) and Theorem 1 in [5] that $D_1(t)$ (resp. $D_2(t)$) is stochastically *increasing* (resp. *decreasing*) with respect to $\underline{\lambda}_1$.

Let us now show that $B_2(t)$ is stochastically increasing in $\underline{\lambda}_1$. We have:

$$Y_2(t) = y_2 + A_2(t) - C_2(t) - E_2(t); \tag{2.7}$$

$$Y_2'(t) = y_2 + A_2(t) - C_2'(t) - E_2'(t), \tag{2.8}$$

for all $t \geq 0$, where

- $\{A_2(t),\, t \geq 0\}$ is a Poisson process with rate λ_2;
- $C_2(t)$ (resp. $C_2'(t)$) is the number of *potential* (T3) transitions that did not fire (i.e. rejected calls) for $\mathbf{Y}_2$ (resp. $\mathbf{Y}_2'$) by time t.

From (2.7)-(2.8) we get

$$C_2'(t) - C_2(t) = [Y_2(t) - Y_2'(t)] + [E_2(t) - E_2'(t)], \tag{2.9}$$

for all $t \geq 0$.

The first term in the right-hand side of (2.9) is nonnegative from (2.3). The second term in the right-hand side of (2.9) is also nonnegative (see (2.6)). So,

$$C_2(t) \leq C_2'(t), \tag{2.10}$$

for all $t \geq 0$.

Noting now that $\{C_2(t),\, t \geq 0\}$ has the same distribution as $\{B_2(t),\, t \geq 0\}$ in $[\underline{\lambda}_1, \lambda_2]$, and that $\{C_2'(t),\, t \geq 0\}$ as the same distribution as the number of type-2 rejected calls in $[\underline{\lambda}_1', \lambda_2]$, we conclude with (2.10) and Theorem 1 in [5] that $B_2(t)$ is stochastically increasing in $\underline{\lambda}_1$. ∎

To avoid unnecessary complications, and also because this is the only case of interest in the context of circuit-switched networks, we shall assume in the remainder of this section that $\mu_k(n) := n\mu_k$, for $n = 1, 2, \ldots, \lfloor N/b_k \rfloor$ and $k = 1, 2$.

Many monotonicity results for steady-state performance measures can be derived from Theorem 2.1. These results are collected in Corollary 2.1.

Define $\underline{\rho}_k := (\rho_k(0), \rho_k(1), \ldots, \rho_k(\lfloor N/b_k \rfloor - 1))$, where $\rho_k(\cdot) := \lambda_k(\cdot)/\mu_k$, $k = 1, 2$.

Corollary 2.1 *Assume that the call holding-time distributions are arbitrary. Then:*

1. *$E(f(X_1))$ is increasing (resp. decreasing) in $\underline{\lambda}_1$ and in $\underline{\rho}_1$ (resp. μ_1) for all increasing mappings $f : \mathbb{R} \to \mathbb{R}$;*

2. *$E(f(X_2))$ is decreasing (resp. increasing) in $\underline{\lambda}_1$ and in $\underline{\rho}_1$ (resp. μ_1) for all increasing mappings $f : \mathbb{R} \to \mathbb{R}$;*

3. *T_1 is increasing in $\underline{\lambda}_1$;*

4. *T_2 is decreasing (resp. increasing) in $\underline{\lambda}_1$ and in $\underline{\rho}_1$ (resp. μ_1);*

5. *β_2 is increasing (resp. decreasing) in $\underline{\lambda}_1$ and in $\underline{\rho}_1$ (resp. μ_1).*

The same results hold when indices 1 and 2 are interchanged.

Proof. Assume first that the holding times are exponentially distributed, and consider X_k, β_k and T_k ($k = 1, 2$) as functions of $\underline{\lambda}_1$. Then, statements 1-5 directly follow from Theorem 2.1.

Noting now that $\mathbf{X}$ depends on $\lambda_1(\cdot)$ and μ_1 only through $\rho_1(\cdot)$, cf. (1.1)-(1.2), we immediately get the extensions of statements 1, 2, 4, 5 to the parameters $\underline{\rho}_1$ and μ_1 (use also (1.5) and (1.4) for 4 and 5, respectively).

Finally, the validity of Corollary 2.1 for arbitrary call holding-time distributions is a consequence of the *insensitivity* of the distribution of $\mathbf{X}$ to the call holding-time distributions. ∎

Many questions remain open at this point. In particular, does $\underline{\lambda}_k \to \mathrm{E}(f(Z))$ have a monotonic behavior for $k = 1,2$ if $K = 2$? Does $\underline{\lambda}_k \to \beta_k$ (resp. $\mu_k \to T_k$) have a monotonic behavior for $k = 1,2$ if $K = 2$? We shall see in the next section that the answers to these questions are negative in general. For $K = 2$, we shall show that the answer to the first question (and therefore to the second ones if $K = 2$) is positive, provided some additional restrictions are put on b_1 and b_2. Some results will also be obtained for the case $K > 2$.

Remark 2.1 Unfortunately, for $K > 2$ we have been unable to extend the coupling technique used in the proof of Theorem 2.1.

3 Monotonicity: Steady-State Analysis

We assume throughout this section that the call holding-times have *general* distributions and that $\mu_k(X_k) := \mu_k X_k$ for $k = 1,2,\ldots,K$. Define $\underline{\rho}_k := (\rho_k(0), \rho_k(1), \ldots, \rho_k(\lfloor N/b_k \rfloor - 1))$, where $\rho_k(\cdot) := \lambda_k(\cdot)/\mu_k$ for $k = 1,2,\ldots,K$.

As mentioned earlier, the presence of the normalization constant $G(N)$ in (1.1) is a major obstacle in the search for qualitative properties. A similar problem was encountered by Shanthikumar and Yao in their qualitative study of product-form closed queuing networks [13]. They nevertheless got over this difficulty by using the notion of *likelihood-ratio ordering* for random variables. We shall show in this section that this approach also yields interesting results in our case.

We first recall the definition of the likelihood-ratio ordering (see [6] and [12] for further information).

Definition 3.1 Let U and V be two random variables with common support $\{x_0, x_1, \ldots, x_M\}$, where $x_n < x_{n+1}$ for $0 \le n \le M-1$. We say that U is smaller than V in the sense of *likelihood-ratio ordering*, and write $U \le_{lr} V$, if

$$\frac{\mathrm{P}(U = x_n)}{\mathrm{P}(U = x_{n+1})} \ge \frac{\mathrm{P}(V = x_n)}{\mathrm{P}(V = x_{n+1})},$$

for all $n = 0,1,\ldots,M-1$.

It is known that $U \le_{lr} V$ implies $U \le_{st} V$ (stochastic ordering) [12]. A very short proof of this result specialized to the case of discrete random variables is given in Appendix A.

For $k = 1,2,\ldots,K$ define:

$$Z_{(k)} \;:=\; \sum_{1 \le l \ne k \le K} b_l X_l;$$

$$g_k(n) := \frac{\prod_{l=0}^{n-1} \lambda_k(l)}{n!\, \mu_k^n};$$

$$\mathbf{b} \cdot \mathbf{n}_{(k)} := \sum_{1 \le l \ne k \le K} b_l n_l.$$

We have the following theorem:

Theorem 3.1 *With respect to $\underline{\lambda}_k$, X_k is increasing and $Z_{(k)}$ is decreasing in the sense of likelihood-ratio ordering for $k = 1, 2, \ldots, K$.*

Proof. From (1.1) we get

$$\frac{\mathrm{P}(X_k = n)}{\mathrm{P}(X_k = n+1)} = \frac{(n+1)\,\mu_k}{\lambda_k(n)} \times \frac{G_{(k)}(N - nb_k)}{G_{(k)}(N - (n+1)b_k)}, \tag{3.1}$$

where

$$G_{(k)}(x) := \sum_{\mathbf{b}\cdot\mathbf{n}_{(k)} \le x} \prod_{1 \le l \ne k \le K} g_l(n_l).$$

Since $G_{(k)}(N - l\,b_k)$ $(0 \le l \le \lfloor N/b_k \rfloor)$ does not depend on $\underline{\lambda}_k$, it follows from (3.1) that X_k is increasing in $\underline{\lambda}_k$ in terms of the likelihood-ratio ordering.

Let $\{x_0, x_1, \ldots, x_M\}$ be the support of $Z_{(k)}$. Recall that $x_n < x_{n+1}$ for $n = 0, 1, \ldots, M-1$. Then, cf. (1.1),

$$\frac{\mathrm{P}(Z_{(k)} = x_n)}{\mathrm{P}(Z_{(k)} = x_{n+1})} = \frac{H_{(k)}(x_n)}{H_{(k)}(x_{n+1})} \times \frac{\displaystyle\sum_{b_k n_k \le N - x_n} g_k(n_k)}{\displaystyle\sum_{b_k n_k \le N - x_{n+1}} g_k(n_k)}, \tag{3.2}$$

where

$$H_{(k)}(x) := \sum_{\mathbf{b}\cdot\mathbf{n}_{(k)} = x} \prod_{1 \le l \ne k \le K} g_l(n_l).$$

Since $H_{(k)}(x_l)$ $(0 \le l \le M)$ does not depend on $\underline{\lambda}_k$, it suffices to show that the second term on the right-hand side of (3.2) is decreasing in $\underline{\lambda}_k$.

Define $a := \lfloor (N - x_{n+1})/b_k \rfloor$ and $b := \lfloor (N - x_n)/b_k \rfloor$ $(a < b)$. Let W be a random variable with support $\{0, 1, \ldots, b\}$ such that

$$\mathrm{P}(W = j) = \mathrm{P}(W = 0)\, g_k(j), \quad 0 \le j \le b.$$

Since W is clearly increasing in $\underline{\lambda}_k$ in the sense of likelihood-ratio ordering, this implies that $\mathrm{E}(f(W))$ is increasing in $\underline{\lambda}_k$ for all increasing mappings $f : \mathbb{R} \to \mathbb{R}$. In particular,

$$\mathrm{P}(W \leq a) = \frac{\sum_{j \leq a} g_k(j)}{\sum_{j \leq b} g_k(j)}$$

is increasing in $\underline{\lambda}_k$, which concludes the proof. ∎

Corollary 3.1 *For* $k = 1, 2, \ldots, K$*:*

1. X_k *is increasing (resp. decreasing) w.r.t.* $\underline{\rho}_k$ *(resp.* μ_k*) for the likelihood-ratio ordering;*
2. $Z_{(k)}$ *is decreasing (resp. increasing) w.r.t.* $\underline{\rho}_k$ *(resp.* μ_k*) for the likelihood-ratio ordering;*
3. T_k *is increasing w.r.t.* $\underline{\lambda}_k$*;*
4. *For* $K = 2$*,* T_k *is decreasing (resp. increasing) w.r.t.* $\underline{\lambda}_l$ *and* $\underline{\rho}_l$ *(resp.* μ_l*) for* $1 \leq k \neq l \leq 2$.

Remark 3.1 Theorem 3.1 and Corollary 3.1 extend the statements 1-4 of Corollary 2.1 both to $K > 2$ and to arbitrary vector arrival rates (i.e. the $\lambda_k(\cdot)$'s need not be increasing functions, $k = 1, 2, \ldots, K$).

Let $\rho_k := \lambda_k/\mu_k$, $k = 1, 2, \ldots, K$. The next theorem partially answers the questions listed at the end of Section 2. We assume from now on that $K = 2$, unless otherwise stated.

Theorem 3.2 *Assume* $b_1 \leq b_2 \leq N$ *and* $\lambda_k(\cdot) := \lambda_k$, $k = 1, 2$. *Then,* Z *the total number of calls in progress in* $[\lambda_1, \lambda_2]$ *is increasing as a function of* λ_2 *in the sense of likelihood-ratio ordering if and only if* b_2/b_1 *is an integer or* $\lfloor N/b_1 \rfloor b_1 \leq b_2$.

Proof. Take $\lambda_2' > \lambda_2$ and define Z' as the total number of calls in progress in the system $[\lambda_1, \lambda_2']$. Also introduce for $0 \leq n \leq N$,

$$q_n := \mathrm{P}(Z = n) \quad \text{and} \quad q_n' := \mathrm{P}(Z' = n).$$

It has been shown by Kaufman [7] and Roberts [10] that

$$q_n = \frac{1}{n}\left(b_1 \rho_1 \, q_{n-b_1} + b_2 \rho_2 \, q_{n-b_2}\right), \tag{3.3}$$

for $n = 1, \dots, N-1$, with $q_n = 0$ if $n < 0$.

Sufficient condition. Assume first that b_2/b_1 is an integer. We can assume without loss of generality that $b_1 = 1$. From the recursion (3.3) we deduce that the set of inequalities,

$$\frac{q_j}{q_{j+1}} \geq \frac{q'_j}{q'_{j+1}}; \tag{3.4}$$

$$\rho_2 \frac{q_{j+2-b_2}}{q_{j+1}} \leq \rho'_2 \frac{q'_{j+2-b_2}}{q'_{j+1}}, \tag{3.5}$$

is satisfied for $0 \leq j < N$, where $\rho'_2 := \lambda'_2/\mu_2$.

Note that (3.4) shows that Z is increasing for the likelihood-ratio ordering w.r.t. λ_2 (cf. Definition 2.1).

The proof proceeds by induction. Using (3.3) we first check that (3.4) and (3.5) both hold true for $j = 0$. Assume that (3.4) and (3.5) are still true for $j \leq n-1$. Then, cf. (3.3),

$$\frac{q_{n+1}}{q_n} = \frac{1}{n+1}\left[\rho_1 + b_2\rho_2 \frac{q_{n+1-b_2}}{q_n}\right] \leq \left[\rho_1 + b_2\rho'_2 \frac{q'_{n+1-b_2}}{q_n}\right] = \frac{q'_{n+1}}{q'_n},$$

from the induction hypothesis on (3.5).

On the other hand,

$$\begin{aligned} \frac{q_{n+1}}{\rho_2\, q_{n+2-b_2}} &= \frac{1}{n+1}\left[\frac{\rho_1}{\rho_2}\cdot\frac{q_n}{q_{n+2-b_2}} + b_2\,\frac{q_{n+1-b_2}}{q_{n+2-b_2}}\right] \\ &\geq \frac{1}{n+1}\left[\frac{\rho_1}{\rho_2}\cdot\frac{q_n}{q_{n+2-b_2}} + b_2\,\frac{q'_{n+1-b_2}}{q'_{n+2-b_2}}\right] \quad \text{(induction hypothesis on (3.4))} \\ &\geq \frac{1}{n+1}\left[\frac{\rho_1}{\rho'_2}\cdot\frac{q'_n}{q'_{n+2-b_2}} + b_2\,\frac{q'_{n+1-b_2}}{q'_{n+2-b_2}}\right] = \frac{q'_{n+1}}{\rho'_2\, q'_{n+2-b_2}}, \end{aligned}$$

since

$$\frac{q_n}{\rho_2\, q_{n+2-b_2}} = \frac{q_{n+1-b_2}}{q_{n+2-b_2}}\cdot\frac{q_n}{\rho_2\, q_{n+1-b_2}} \geq \frac{q'_{n+1-b_2}}{q'_{n+2-b_2}}\cdot\frac{q'_n}{\rho'_2\, q'_{n+1-b_2}} = \frac{q'_n}{\rho'_2\, q'_{n+2-b_2}},$$

from the induction hypothesis on (3.4) and (3.5).

Let us now assume that b_2/b_1 is not an integer and that $\lfloor N/b_1 \rfloor b_1 \leq b_2$. This implies that the support of Z and Z' is $\{0, b_1, \dots, \alpha\, b_1, b_2\}$, where $\alpha := \lfloor N/b_1 \rfloor$.

We have, cf. (1.1),

$$\frac{q_{j\,b_1}}{q_{(j+1)\,b_1}} = \frac{\mathrm{P}\,(X_1 = j, X_2 = 0)}{\mathrm{P}\,(X_1 = j+1, X_2 = 0)} = \frac{j+1}{\rho_1}, \tag{3.6}$$

for $j = 0, 1, \ldots, \alpha - 1$. Since the rightmost term in (3.6) does not depend on λ_2, we obviously have

$$\frac{q_{j\,b_1}}{q_{(j+1)\,b_1}} = \frac{q'_{j\,b_1}}{q'_{(j+1)\,b_1}},$$

for $j = 0, 1, \ldots, \alpha - 1$. Moreover,

$$\frac{q_{\alpha\,b_1}}{q_{b_2}} = \frac{\mathrm{P}\,(X_1 = \alpha, X_2 = 0)}{\mathrm{P}\,(X_1 = 0, X_2 = 1)} = \frac{\rho_1^\alpha}{\rho_2\,\alpha!} \geq \frac{\rho_1^\alpha}{\rho_2'\,\alpha!} = \frac{\mathrm{P}\,(X_1 = \alpha, X_2' = 0)}{\mathrm{P}\,(X_1 = 0, X_2' = 1)} = \frac{q'_{\alpha b_1}}{q'_{b_2}},$$

which shows that Z is increasing as a function of λ_2 in the sense of likelihood-ratio ordering.

Necessary condition. Assume that (*i*) b_2/b_1 is not an integer and (*ii*) $\lfloor (N/b_1 \rfloor b_1 > b_2$. Let $\{x_0, x_1, \ldots, x_M\}$ be the support of Z and Z'. Let k be the integer such that

$$x_k = b_2. \tag{3.7}$$

We show that

$$\frac{q_{x_{k-1}}}{q_{x_k}} > \frac{q'_{x_{k-1}}}{q'_{x_k}} \tag{3.8}$$

and

$$\frac{q_{x_k}}{q_{x_{k+1}}} < \frac{q'_{x_k}}{q'_{x_{k+1}}}, \tag{3.9}$$

so that $Z \not\leq_{lr} Z'$ under conditions (i) and (ii).

We first observe that

$$\begin{aligned} x_j &= jb_1, \quad \text{for } j = 0, 1, \ldots, k-1; & (3.10)\\ x_{k+1} &= kb_1, \quad \text{(condition (ii))}. & (3.11) \end{aligned}$$

Again, using the recursion (3.3) along with (3.7), (3.10), we get

$$\frac{q_{x_k}}{q_{x_{k-1}}} = \frac{1}{b_2}\left[b_1\rho_1\,\frac{q_{b_2-b_1}}{q_{(k-1)b_1}} + b_2\rho_2\,\frac{q_0}{q_{(k-1)b_1}}\right]. \tag{3.12}$$

But $q_{b_2-b_1} = 0$ ($b_2 - b_1$ is not a feasible state from condition (i)) and clearly $q_0/q_{(k-1)b_1}$ does not depend on λ_2. Consequently, cf. (3.12),

$$\frac{q_{x_k}}{q_{x_{k-1}}} < \frac{q'_{x_k}}{q_{x'_{k-1}}},$$

which is exactly the inequality (3.8).

On the other hand,

$$\begin{aligned}
\frac{q_{x_{k+1}}}{q_{x_k}} &= \frac{1}{x_{k+1}}\left[b_1\rho_1\frac{q_{x_{k+1}-b_1}}{q_{x_k}} + b_2\rho_2\frac{q_{x_{k+1}-b_2}}{q_{x_k}}\right] \\
&= \frac{1}{x_{k+1}}\left[b_1\rho_1\frac{q_{x_{k-1}}}{q_{x_k}} + b_2\rho_2\frac{q_{kb_1-b_2}}{q_{x_k}}\right] \quad \text{(cf. (3.10) and (3.11))} \\
&= \frac{b_1\rho_1}{x_{k+1}}\cdot\frac{q_{x_{k-1}}}{q_{x_k}}, \qquad (3.13)
\end{aligned}$$

since $kb_1 - b_2$ is not a feasible state (condition (i)). Now using now (3.8) in (3.13) gives us (3.9), which concludes the proof. ∎

Corollary 3.2 *If $b_2/b_1 \in \mathbb{N}$ or $\lfloor N/b_1 \rfloor b_1 \le b_2$ then:*

1. *Z is increasing (resp. decreasing) for the likelihood-ratio ordering w.r.t. ρ_2 (resp. μ_2);*
2. *$E(f(Z))$ is increasing (resp. decreasing) w.r.t. λ_2 and ρ_2 (resp. μ_2) for all increasing mappings $f : \mathbb{R} \to \mathbb{R}$;*
3. *β_2 is increasing (resp. decreasing) w.r.t. λ_2 and ρ_2 (resp. μ_2);*
4. *T_2 is increasing w.r.t. μ_2.*

Proof. Statements 1 and 2 are obvious. Statement 3 follows from statement 2 by choosing $f(x) = \mathbf{1}(x > N - b_2)$, cf. (1.4). Statement 4 follows from statement 3 and (1.6). ∎

Corollary 3.3 *If $K \ge 2$, $b := b_k$ $(1 \le k \le K-1)$ and $b \le b_K \le N$, then Z is increasing (resp. decreasing) w.r.t. λ_K and ρ_K (resp. μ_K) in the sense of likelihood-ratio ordering if and only if b_K/b is an integer or $\lfloor N/b \rfloor b \le b_K$.*

Proof. For $K \ge 2$, Kaufman-Roberts' recursion reads:

$$q_n = \frac{1}{n}\sum_{k=1}^{K} b_k\rho_k q_{n-b_k}.$$

Therefore, the proof is analogous to the proof of Theorem 3.1 provided that ρ_1 is replaced by $\sum_{k=1}^{K-1}\rho_k$. ∎

Since the stochastic ordering is weaker than the likelihood-ratio ordering, we may want to know whether Z is also increasing for the *stochastic ordering* w.r.t. ρ_2 if b_2/b_1 is not an integer and if

$\lfloor N/b_1 \rfloor b_1 > b_2$. Also, we may want to know whether the mapping $\rho_2 \to \beta_2$ is increasing under the aforementioned conditions, in the case where the answer to the first question is negative. The examples below show that the answer is no in both cases.

Example 1.
$b_1 = 2$, $b_2 = 3$, $N = 4$. Then, Z is stochastically increasing w.r.t. ρ_2 (hint: use (1.1)).

Example 2.
$b_1 = 2$, $b_2 = 3$, $N = 7$. Then, Z is not stochastically increasing w.r.t. ρ_2 (q_7 is not increasing). However, β_2 is increasing w.r.t. ρ_2.

Example 3.
$b_1 = 3$, $b_2 = 5$, $N = 10$. Then, β_2 is not increasing w.r.t. ρ_2 (for large ρ_1, $\partial\beta_2/\partial\rho_2 < 0$ for $\rho_2 = 0$).

The last point we would like to discuss is concerned with the monotonicity of β_1 w.r.t. ρ_1. The example below shows that this function is not monotone in general, even if b_2/b_1 is an integer.

Example 4.
$b_1 = 1$, $b_2 = 2$, $N = 2$. For fixed $\rho_2 > 0$, β_1 is decreasing in $[0, -1 + \sqrt{1 + 2\rho_2})$ and increasing in $(-1 + \sqrt{1 + 2\rho_2}, +\infty)$ w.r.t. ρ_1.

Finally, since $T_1 = \lambda_1(1 - \beta_1)$, Example 4 implies that T_1 is not monotone in general w.r.t. μ_1.

All the monotonicity properties that have been discovered throughout Sections 2 and 3 are collected in Table 1. More steady-state results can be found in [11] for $K > 2$ (the method used by Ross and Yao in their paper is similar to the method employed in [13]).

4 Concavity

We consider the original Erlang blocking model ($K = 1$ and $b_1 = 1$). Denote by λ the call arrival rate and by $1/\mu$ the average call holding-time. Let $X(t)$ be the number of ongoing transmissions at time t. We have the following result:

Theorem 4.1 *For all increasing mappings $f : \mathbb{R} \to \mathbb{R}$ and for all $t > 0$, $E(f(X(t)))$ is concave increasing as a function of λ .*

Proof. The proof is similar to the proof of Theorem 2.1. Let $(\lambda_i)_{1 \le i \le 4}$ be four real numbers such that $0 < \lambda_1 < \lambda_2$, $\lambda_3 < \lambda_4$ and $\lambda_1 < \lambda_3$. We furthermore assume that $\lambda_2 - \lambda_1 = \lambda_4 - \lambda_3$.

We construct four birth-death processes $\mathbf{Y}_i := \{Y_i(t),\, t \ge 0\}$, $i = 1, 2, 3, 4$, on a common probability space by generating transitions using a single Poisson process with rate

$$\gamma := \lambda_4 + \mu N. \tag{4.1}$$

Time-dependent behavior ($K = 2$)

Exponential call holding-times

$\lambda_1(\cdot)$, $\mu_1(\cdot)$, $\mu_2(\cdot)$ increasing functions; $\lambda_2(\cdot)=$ constant

$X_1(t) \uparrow_{st}$ w.r.t. $\underline{\lambda}_1$

$X_2(t) \downarrow_{st}$ w.r.t. $\underline{\lambda}_1$

$B_2(t) \uparrow_{st}$ w.r.t. $\underline{\lambda}_1$

$D_1(t) \uparrow_{st}$ w.r.t. $\underline{\lambda}_1$

$D_2(t) \downarrow_{st}$ w.r.t. $\underline{\lambda}_1$

(The above results hold whenever indices 1 and 2 are interchanged)

Steady-state behavior ($K = 2$)

Arbitrary call holding-times with constant rates

$\lambda_k(\cdot)$ arbitrary function ($k = 1, 2$)

$T_1 \downarrow$ w.r.t. $\underline{\lambda}_2$ (resp. $\underline{\rho}_2$)

$T_1 \uparrow$ w.r.t. μ_2

$\beta_1 \uparrow$ w.r.t. $\underline{\lambda}_2$ (resp. $\underline{\rho}_2$) if $\lambda_2(\cdot)$ is increasing

$\beta_1 \downarrow$ w.r.t. μ_2 if $\lambda_2(\cdot)$ is increasing

(The above results hold whenever indices 1 and 2 are interchanged)

If constant arrival rates and $b_1 \leq b_2 \leq N$ then:

$Z \uparrow_{lr}$ w.r.t. λ_2 (resp. ρ_2) *iif.* $b_2/b_1 \in \mathbf{N}$ or $\lfloor N/b_1 \rfloor b_1 \leq b_2$ $\quad (\dagger)$

$Z \downarrow_{lr}$ w.r.t. μ_2 *iif.* $(\dagger)$

$\beta_2 \uparrow$ w.r.t. λ_2 and ρ_2 *if* $(\dagger)$

$\beta_2 \downarrow$ w.r.t. μ_2 *if* $(\dagger)$

$T_2 \uparrow$ w.r.t. μ_2 *if* $(\dagger)$

Steady-state behavior ($K \geq 2$)

Arbitrary call holding-times with constant rates

$\lambda_k(\cdot)$ arbitrary function ($1 \leq k \leq K$)

$X_k \uparrow_{lr}$ w.r.t. $\underline{\lambda}_k$ (resp. $\underline{\rho}_k$)

$X_k \downarrow_{lr}$ w.r.t. μ_k

$Z_{(k)} \downarrow_{lr}$ w.r.t. $\underline{\lambda}_k$ (resp. $\underline{\rho}_k$)

$Z_{(k)} \uparrow_{lr}$ w.r.t. μ_k

$T_k \uparrow$ w.r.t. $\underline{\lambda}_k$

If constant arrival rates, $b_k := b$ ($1 \leq k \leq K-1$) and $b \leq b_K \leq N$, then:

$Z \uparrow_{lr}$ w.r.t. λ_K (resp. ρ_K) *iff.* $b_K/b \in \mathbf{N}$ or $\lfloor N/b \rfloor b \leq b_K$ $\quad (\ddagger)$

$Z \downarrow_{lr}$ w.r.t. μ_K *iff.* $(\ddagger)$

Table 1: Summary of the monotonicity results

Suppose a point in this Poisson process occurs at time t. Then, at time t,

(T1) with probability λ_1/γ a birth occurs

- in $\mathbf{Y}_i$ if $Y_i(t-) < N$ for $i = 1, 2, 3, 4$;

(T2) with probability $(\lambda_2 - \lambda_1)/\gamma$ a birth occurs

- in $\mathbf{Y}_i$ if $Y_i(t-) < N$ for $i = 2, 4$;

(T3) with probability $(\lambda_3 - \lambda_1)/\gamma$ a birth occurs

- in $\mathbf{Y}_i$ if $Y_i(t-) < N$ for $i = 3, 4$;

(T4) with probability $\mu Y_1(t-)/\gamma$ a death occurs

- in $\mathbf{Y}_i$ for $i = 1, 2, 3, 4$;

(T5) with probability $\mu[Y_2(t-) + Y_3(t-) - Y_1(t-) - Y_4(t-)]/\gamma$ a death occurs

- in $\mathbf{Y}_i$ for $i = 2, 3, 4$;

(T6) with probability $\mu[Y_4(t-) - \min(Y_2(t-), Y_3(t-))]/\gamma$ a death occurs

- in $\mathbf{Y}_i$ for $i = M_t, 4$;

(T7) with probability $\mu[Y_4(t-) - \max(Y_2(t-), Y_3(t-))]/\gamma$ a death occurs

- in $\mathbf{Y}_i$ for $i = m_t, 4$;

(T8) with probability $\mu(N - Y_4(t-))/\gamma$ no event occurs in $\mathbf{Y}_i$, $i = 1, 2, 3, 4$,

where $m_t, M_t \in \{2, 3\}$, $m_t \neq M_t$, are defined by

$$m_t := \arg\min(X_2(t-), X_3(t-)); \tag{4.2}$$

$$M_t := \arg\max(X_2(t-), X_3(t-)), \tag{4.3}$$

for all $t > 0$.

The above construction together with the initial conditions $Y_i(0) = y$, $i = 1, 2, 3, 4$, entails the following inequalities:

$$Y_1(t) + Y_4(t) \leq Y_2(t) + Y_3(t); \tag{4.4}$$

$$Y_1(t) \leq \min(Y_2(t), Y_3(t)) \leq \max(Y_2(t), Y_3(t)) \leq Y_4(t), \tag{4.5}$$

for all $t \geq 0$. It should be noted that (4.4)-(4.5) fully validate the transitions (T5), (T6), (T7).

The proof proceeds by induction. Note first that (4.4)-(4.5) hold for $t = 0$. Let $t > 0$ be a point of the Poisson process and let us assume that (4.4)-(4.5) hold in $[0, t)$.

Assume that one of the transitions (T1)-(T3) occurs at time t (birth transition). Then, by considering successively the five following cases:

(1) $Y_4(t-) < N$;

(2) $\max(Y_2(t-), Y_3(t-)) < N = Y_4(t-)$;

(3) $\min(Y_2(t-), Y_3(t-)) < N = \max(Y_2(t-), Y_3(t-)) = Y_4(t-)$;

(4) $Y_1(t-) < N = Y_2(t-) = Y_3(t-) = Y_4(t-)$;

(5) $Y_1(t-) = N$,

it is readily seen that (4.4)-(4.5) still hold at time t.

Assume now that one of the events (T4)-(T7) (death transition) occurs at time t. Consider the transition (T6). Then, cf. (T6),

$$\min(Y_2(t-), Y_3(t-)) < Y_4(t-). \tag{4.6}$$

Since (4.6) implies that $Y_2(t-) + Y_3(t-) > 0$ (otherwise (4.4) would imply, in particular, that $Y_4(t-) = 0$, which would contradict (4.6)), we have

$$\begin{aligned} Y_1(t) + Y_4(t) &= Y_1(t-) + Y_4(t-) - 1; \\ Y_2(t) + Y_3(t) &= Y_2(t-) + Y_3(t-) - 1, \end{aligned} \tag{4.7}$$

which shows that (4.4) still holds at time t under the transition of type (T6).

In order to show that (4.5) also holds at time t, we have to examine the following two cases:

(i) $Y_1(t-) \leq \min(Y_2(t-), Y_3(t-)) < \max(Y_2(t-), Y_3(t-)) \leq Y_4$;

(ii) $Y_1(t-) \leq \min(Y_2(t-), Y_3(t-)) \leq \max(Y_2(t-), Y_3(t-)) < Y_4$.

In case (i), the inequality (4.5) clearly propagates to time t under the transition of type (T6) . In case (ii), we may have a problem if $Y_1(t-) = \min(Y_2(t-), Y_3(t-)) = \max(Y_2(t-), Y_3(t-)) < Y_4(t-)$, or, equivalently, if $Y_1(t-) = Y_2(t-) = Y_3(t-) < Y_4(t-)$. However, this situation cannot occur since it would contradict the induction hypothesis (4.4). The remaining transitions (T4), (T5), (T7) can be treated in a similar way.

These results are used as follows. First, we observe that the above construction implies that $\mathbf{Y}_i$ has the same distribution as $\{X_i(t),\, t \geq 0\}$, where $X_i(t)$ denotes the number of transmissions in progress at time t when the arrival rate is λ_i, $i = 1, 2, 3, 4$.

Secondly, we apply the coupling theorem in [5] to (4.4)-(4.5), which gives us

$$X_1(t) + X_4(t) \;\leq_{st}\; X_2(t) + X_3(t); \tag{4.8}$$

$$X_1(t) \leq_{st} \min(X_2(t), X_3(t)) \;\leq_{st}\; \max(X_2(t), X_3(t)) \leq_{st} X_4(t), \tag{4.9}$$

for all $t \geq 0$.

The inequality (4.9) shows, in particular, that $X(t)$ is stochastically increasing in λ for all $t > 0$ (note that this property also follows from Theorem 2.1 by setting $\lambda_2 = 0$). By combining this result with the inequality (4.8) we get that $\lambda \rightarrow \mathrm{E}\left(f(X(t))\right)$ is concave increasing for all increasing mappings $f : \mathbb{R} \rightarrow \mathbb{R}$ and all $t > 0$. ∎

Corollary 4.1 *Let X be the number of ongoing transmissions in steady-state. Then, $E(f(X))$ is concave increasing as a function of λ for all increasing mappings $f : \mathbb{R} \rightarrow \mathbb{R}$. In particular, $E(X)$, the blocking probability β ($\beta = P(X > N-1)$) and the throughput T ($T = \mu E(X)$) are all concave and increasing in λ.*

Suppose now that the arrival rate depends on the state of the system. Let $\lambda(i)$ be the arrival rate when there are i transmissions in progress, $i = 0, 1, \ldots, N-1$. Then, by proceeding as in the proof of Theorem 4.1, it can be shown that:

Theorem 4.2 *For all increasing mappings $f : \mathbb{R} \rightarrow \mathbb{R}$ and for all $t > 0$, $E(f(X(t)))$ is concave increasing as a function of $\lambda(i)$, for $i = 0, 1, \ldots, N-1$,*

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A Appendix

We give below a short proof of the fact that

$$U \leq_{lr} V \Longrightarrow U \leq_{st} V,$$

when U and V are discrete random variables with support $\{x_0, x_1, \ldots, x_M\}$.

Assume there exists (at least one) $n \in \{0, 1, \ldots, M\}$ such that $\mathrm{P}(U = x_n) \neq \mathrm{P}(V = x_n)$ (otherwise the result is obviously true), and define

$$l := \min\{n \in \{0, 1, \ldots, M\} : \mathrm{P}(U = x_n) < \mathrm{P}(V = x_n)\}.$$

Note that l is well-defined, since the case where $\mathrm{P}(U = x_n) \geq \mathrm{P}(V = x_n)$ for all $n = 0, 1, \ldots, M$ is impossible (this would imply that $\mathrm{P}(U = x_n) = \mathrm{P}(V = x_n)$ for all $n = 0, 1, \ldots, M$, since otherwise $\sum_{n=0}^{M} \mathrm{P}(V = x_n) < 1$, but then this would contradict the assumption on the distributions of U and V).

So,

$$\mathrm{P}(U = x_n) \geq \mathrm{P}(V = x_n), \quad \text{for } n \leq l - 1,$$

and

$$\mathrm{P}(U \leq x_n) \geq \mathrm{P}(V \leq x_n), \quad \text{for } n \leq l - 1. \tag{A.1}$$

Now, (A.1) together with the fact that $U \leq_{lr} V$ (cf. definition 3.1) implies that

$$\mathrm{P}(U = x_n) < \mathrm{P}(V = x_n), \quad \text{for } n \geq l,$$

and

$$\mathrm{P}(U < x_n) > \mathrm{P}(V < x_n), \quad \text{for } n \geq l,$$

which completes the proof. ∎

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